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On Steady Long Waves on a Viscous Liquid at Small Reynolds Number

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SUMMARY
The generation of steady surface waves on a viscous liquid flowing down an irregular inclined plane is investigated in the shallow-liquid approximation. A non-linear differential equation gives the surface elevation and a numerical solution is presented for a periodic two-dimensional flow. Linearisation of this equation enables three-dimensional small-amplitude disturbances to be considered.

1. Introduction

In an earlier paper [1] by the author, an analysis was given of steady waves generated on the surface of a viscous liquid by perturbations in an inclined plane down which the liquid was flowing. The basic shear flow is one of the simplest instances in which steady flow can be maintained with a free surface. The equations of motion and the boundary conditions were linearised and the forced waves were analysed from perturbations from the basic shear flow with inertia effects small.

The character of the surface waves depends on three parameters: the Reynolds number R, the angle of inclination β of the plane and the ratio of mean liquid depth and a typical disturbance length ε. In the previous paper the case of small R and unrestricted β and ε was examined. The problem is reconsidered here for effectively zero Reynolds number and small ε. Certain justifiable approximations can be made which result in some simplification in the boundary-value problems. Furthermore, three-dimensional disturbances can be included in the analysis.

Figure 1 shows the coordinate scheme, the forcing wave $y=q(x, z)$ and the free surface $y=h(x, z)$. The surface $y=q(x, z)$ is a perturbation from a plane inclined at the angle β to the

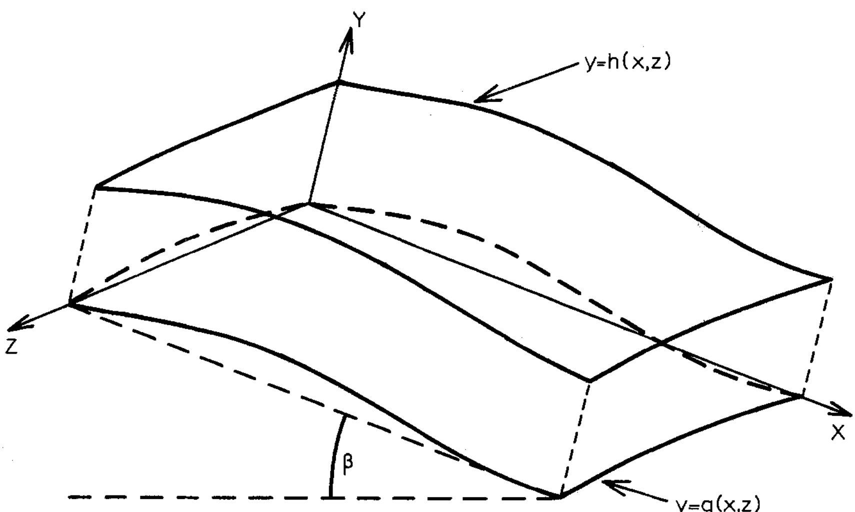

Figure 1.

horizontal. Suppose h_0 is the mean depth of liquid, q_0 the maximum amplitude of $q(x, z)$, x_0 a typical length of the disturbance down the plane and z_0 a typical length across the inclined plane. For the shallow liquid theory we assume that q_0 and h_0 are both small compared with x_0 and z_0, but may themselves be of the same order. In the latter case a nonlinear differential equation must be solved for h. If in addition q_0 is small compared with h_0 further linearisation is possible, and the results by this method can be satisfactorily compared with those in [1].

In these flows of high viscosity and low fluid speed the question of stability does not really arise. However the nonlinear wave problem does reveal a steepening of the wave on the downstream side of the crest.

2. Equations and boundary conditions

With inertia terms neglected, the pressure p and the components of fluid velocity (u, v, w) satisfy

$$p_x = \rho g \sin\beta + \rho\nu\nabla^2 u\,, \tag{2.1}$$

$$p_y = -\rho g \cos\beta + \rho\nu\nabla^2 v\,, \tag{2.2}$$

$$p_z = \rho\nu\nabla^2 w\,, \tag{2.3}$$

in a uniform gravitational field. In these equations ρ is the density of the fluid, ν its kinematic viscosity and ∇^2 the usual three-dimensional Laplacian operator. In addition, continuity demands that

$$u_x + v_y + w_z = 0\,. \tag{2.4}$$

Since we are considering a shear layer of fluid and since the wavelength of disturbances is large compared with the mean depth, we may reasonably suppose that

$$\nabla^2 \approx \partial^2/\partial y^2$$

Further, the assumptions $q_0 \ll x_0$, $q_0 \ll z_0$ imply a perturbation of the flow from that down an inclined plane and because of this we may discard y-variations of v compared with those of u and w. Equations (2.1)–(2.3) now approximate to

$$p_x = \rho g \sin\beta + \rho\nu u_{yy}\,, \tag{2.5}$$

$$p_y = -\rho g \cos\beta\,, \tag{2.6}$$

$$p_z = \rho\nu w_{yy}\,, \tag{2.7}$$

for p, u and w, whilst v can then be determined from (2.4).

On $y = q(x, z)$, we impose the usual no-slip conditions:

$$u = v = w = 0\,. \tag{2.8}$$

Continuity of stress is required at the free surface $y = h(x, z)$. The normal to the free surface has components $(h_x, -1, h_z)$ and if we assume that $|h_x|$ and $|h_z|$ are small in comparison with 1, the continuity of stress at $y = h(x, z)$ becomes to the lowest order

$$u_y + v_x = 0\,, \quad p = 2\rho\nu v_y\,, \quad v_z + w_y = 0\,,$$

(see Wehausen [2], p. 574). Again we neglect x- and z-derivatives compared with y-derivatives and drop v_y in the second condition. Thus on $y = h(x, z)$

$$u_y = 0\,, \quad p = 0\,, \quad w_y = 0\,. \tag{2.9}$$

Finally the kinematic surface condition

$$u h_x - v + w h_z = 0 \quad \text{on} \quad y = h(x, z)\,, \tag{2.10}$$

produces the differential equation for h.

The approximate equations and boundary conditions can be deduced by a systematic approximation procedure, but the main assumptions are those explained above. This is a long wave theory in which the whole fluid layer is treated as a boundary layer.

We note that if $q=0$, then $v=w=0$ and

$$u = gy(2h_0 - y)/(2\nu),$$

the usual velocity distribution for a liquid of constant depth h_0 flowing down an inclined plane. As Benjamin [3] remarks, the character of any disturbance to the basic flow depends on the Reynolds number $R = P/\nu$, where $P = 2h_0 u(h_0)/3$ is the rate of volume flow per unit span of the stream. Clearly $R = gh_0^3 \sin\beta/(3\nu^2)$ and we are investigating the case of small R.

3. Solutions of the Equations

From (2.6) and (2.9), we see that

$$p = \rho g(h-y)\cos\beta, \tag{3.1}$$

indicating, as we might perhaps expect, that the pressure is hydrostatic to the lowest order. Substitution of this pressure field into (2.5) and (2.7) yields two ordinary differential equations for u and w subject to boundary conditions (2.8) and (2.9). Their solutions are

$$u = g(y-q)(h_x\cos\beta - \sin\beta)(y+q-2h)/(2\nu), \tag{3.2}$$

$$w = gh_z\cos\beta(y-q)(y+q-2h)/(2\nu), \tag{3.3}$$

Integration of continuity condition (2.4) together with (2.8) gives

$$v = -g(y-q)[(y-q)(h_{xx}+h_{zz})\{\tfrac{1}{3}(y+2q)-h\}\cos\beta + (h_x\cos\beta - \sin\beta) \cdot\{2q_x(h-q)-h_x(y-q)\} + h_z\cos\beta\{2q_z(h-q)-h_z(y-q)\}]/(2\nu) \tag{3.4}$$

Finally, substitution of (3.1), (3.2) and (3.3) into the kinematic surface condition (2.10) produces the following nonlinear differential equation for h:

$$(h_{xx}+h_{zz})(h-q)+3(h_x-\tan\beta)(h_x-q_x)+3h_z(h_z-q_z)=0. \tag{3.5}$$

It is convenient at this stage to non-dimensionalise the terms in (3.5). With $x=x_0X$, $z=x_0Z$, $q(x,z)=q_0Q(X,Z)$ and $h(x,z)=q_0H(X,Z)$, equation (3.5) becomes

$$(H_{XX}+H_{ZZ})(H-Q)+3(H_X-k)(H_X-Q_X)+3H_Z(H_Z-Q_Z)=0, \tag{3.6}$$

where $k = x_0\tan\beta/q_0$.

4. Two-Dimensional Flow

In the absence of variations of Q and H with respect to Z, equation (3.6) reduces to the ordinary differential

$$H_{XX}(H-Q)+3(H_X-k)(H_X-Q_X)=0, \tag{4.1}$$

where Q is a given function of X. This equation may be integrated once to give

$$(H_X-k)(H-Q)^3 = C, \tag{4.2}$$

where C is a constant. Let us now suppose that $0<\beta<\frac{1}{2}\pi$. Since H must exceed Q for each X and H_X must be less than k for some X (this is equivalent to the restriction $h_x<\tan\beta$), we conclude that C is a negative constant and further that the inequality $H_X<k$ is satisfied for all X. This implies that the tangent plane to the surface cannot be horizontal at any point.

The constant C can be interpreted by considering the rate of flow down the plane. The rate of volume flow per unit span through any plane $x=$ constant must be a constant P, say, where

$$P = \int_q^h u\,dy = -(H-Q)^3(H_X-k)gq_0^4\cos\beta/(3\nu x_0). \tag{4.3}$$

Comparison of (4.2) and (4.3) shows that

$$C = -3\nu x_0 P/(g q_0^4 \cos\beta) . \tag{4.4}$$

Equation (4.2) does not seem to have an elementary solution, except in the case $\beta = \frac{1}{2}\pi$, when $1/k \to 0$ and $H = Q + \text{constant}$. In other words in shallow flow down a vertical plate the free surface adopts the same shape as the perturbing wave. Equation (4.2) can be used to obtain inverse solutions; for a given function H, Q can be found without integration. However, the equation can be solved numerically without difficulty using standard techniques.

Suppose that the forcing wave is sinusoidal with equation $q = A\cos\omega x$. In the previous notation we choose $q_0 = A$ and $x_0 = 1/\omega$ so that $Q = \cos X$. Equation (4.2) becomes

$$(H_X - k)(H - \cos X)^3 = C . \tag{4.5}$$

The equation contains two parameters C and k which need to be specified. The values $k = 1/\pi$ and $C = -1/\pi$ were chosen for the computation of a particular solution. This case, in which $C = -k$, seemed interesting since from (4.4)

$$P = A^3 g \sin\beta/(3\nu) .$$

In the flow down an inclined plane with the same volume flow P we observe that $A = h_0$, the

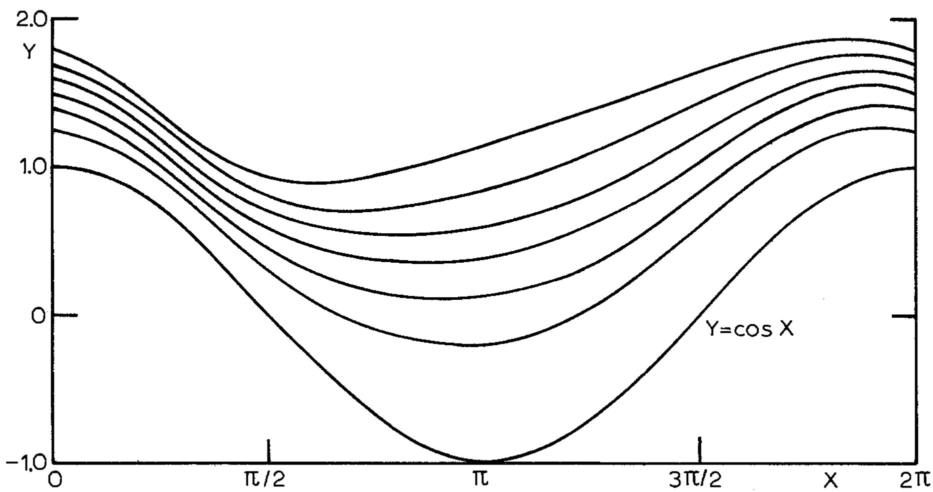

Fig. 2

depth. This, of course, can only be an estimate since we do not know a priori what mean depth will correspond to the solution of (4.5).

Of all the solutions of (4.5) we require the periodic one, that is the solution of (4.5) subject to the condition $H(0) = H(2\pi)$. Some simple estimates establish the approximate position of the isolated periodic solution. A step-by-step integration procedure was used to determine H. The surface wave obtained is shown in figure 2. In the theory of differential equations this periodic solution is unstable since neighbouring solutions diverge from the periodic solution as X increases. This is certainly evident in the numerical scheme and also analytically in the linearised equation which will be considered later in this section. This instability which appears in the differential equation will not correspond to physical instability, and can be accounted for by the absence of the x-derivatives in the equations of motion. If these terms had been included their effect would have been to damp out any waves of increasing amplitude.

The stream function ψ is given by integration of

$$u = \psi_y , \quad v = -\psi_x .$$

Several streamlines are shown in Fig. 2. The surface wave has a characteristic steepness on the downstream side of the crest, together with a shift upstream of the points of maximum and minimum height with respect to the forcing wave.

Equation (4.2) can be linearised if the mean depth is large compared with the perturbation amplitude. If we let $H = H' + h_0/q_0$ where h_0 is the mean depth, and substitute H into (4.2), retaining only constant and first degree terms in H and Q, we find that

$$C = -k(h_0/q_0)^3$$

and

$$H'_X - mH' = -mQ\,, \tag{4.6}$$

where $m = 3kq_0/h_0$. The solution of this equation with the transient ignored can be written as the convolution integral

$$H' = m\mathrm{e}^{mX} \int_X^\infty \mathrm{e}^{-mu} Q(u)\,du\,. \tag{4.7}$$

For a sinusoidal forcing term given by $q = A \cos \omega x$, (4.7) simplifies to

$$H' = m\,\mathrm{e}^{mX} \int_X^\infty \mathrm{e}^{-mu} \cos u\,du$$
$$= \mathscr{R}\{m(m+i)\mathrm{e}^{iX}/(m^2+1)\}\,.$$

With respect to the forcing wave this wave has an amplitude and phase difference given by

$$3A/(q+\varepsilon^2 \cot^2 \beta)^{\frac{1}{2}}\,, \quad \tan^{-1}(\tfrac{1}{3}\varepsilon \cot \beta)\,,$$

where $\varepsilon = \omega h_0$. For small ε, these agree with the amplitude and phase difference derived in equations (5.4) and (5.5) of [1] which were obtained from the full linearised equations. This agreement provides a useful check on the scope of the shallow-liquid approximation developed in the present work.

5. Three-Dimensional Waves

For general disturbances the full equation (3.6) must be considered. This nonlinear partial differential equation has no obvious general solution and as in the previous section we attempt a linearisation procedure. Let $H = H' + h_0/q_0$ in equation (3.6) and discard terms of higher degree than the first in H, Q and their derivatives. We find that H' satisfies

$$H'_{XX} + H'_{ZZ} - mH'_X = -mQ_X\,, \tag{5.1}$$

where, as before, Q is a prescribed function of X and Z. Taking double Fourier transforms of this equation with respect to X and Z in terms of parameters r and s respectively, we find that the transform $\overline{H}'$ of the solution can be expressed as

$$\overline{H}' = mir\,\overline{Q}/(r^2+s^2+mir)\,, \tag{5.2}$$

where $\overline{Q}$ is the double transform of Q according to the definition

$$\overline{Q}(r,s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(X,Z)\mathrm{e}^{-irX-isZ}\,dX\,dZ\,.$$

The inversion theorem applied to (5.2) expresses the free surface elevation as a double Fourier integral:

$$H'(X,Z) = \frac{mi}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{r\overline{Q}\,\mathrm{e}^{iXr+iZs}}{r^2+s^2+mir}\,dr\,ds\,. \tag{5.3}$$

Alternatively, the solution can be obtained by considering the right-hand side of (5.2) as the product of two transforms and expressing it as a double convolution integral along the lines described by Sneddon [4], p. 44. Treating (5.2) as the product of $ir/(r^2+s^2+mir)$ and $m\overline{Q}$, we first determine the function whose double Fourier transform is given by the former expression. This function, $\eta(X, Z)$ say, is

$$\eta = \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{r\,e^{irX+isZ}}{r^2+s^2+mir}\,dr\,ds\,,$$
$$= \frac{i}{2\pi^2}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{r\,e^{irX}\cos(sZ)}{(r+\frac{1}{2}mi)^2+(s^2+\frac{1}{4}m^2)}\,dr\,ds\,, \tag{5.4}$$

by symmetry in s. The substitution $r = R-\frac{1}{2}mi$ transforms (5.4) into

$$\eta = \frac{i\,e^{mX/2}}{2\pi^2}\int_{0}^{\infty}\int_{-\infty+\frac{1}{2}mi}^{\infty+\frac{1}{2}mi}\frac{(R-\frac{1}{2}mi)e^{iRX}\cos(sZ)}{R^2+s^2+\frac{1}{4}m^2}\,dR\,ds\,,$$
$$= \frac{i\,e^{mX/2}}{2\pi^2}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{(R-\frac{1}{2}mi)e^{iRX}\cos(sZ)}{R^2+s^2+\frac{1}{4}m^2}\,dR\,ds\,, \tag{5.5}$$

the deformation of the contour being permissible since the integrand has no singularities on or between the two infinite contours and also since the contribution from the closure of the two contours vanishes in the limit. The integrals with respect to both s and R can be successively read off from tables of Fourier transforms (Erdélyi [5], pp. 8, 65, 16, 17). We find that

$$\eta = m e^{mX/2}[K_0\{\tfrac{1}{2}m(X^2+Z^2)^{\frac{1}{2}}\} - X(X^2+Z^2)^{-\frac{1}{2}}K_1\{\tfrac{1}{2}m(X^2+Z^2)^{\frac{1}{2}}\}]/(4\pi)\,, \tag{5.6}$$

where K_0 and K_1 are modified Bessel functions. Finally the convolution integral for H' becomes

$$H'(X, Z) = \frac{m^2}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{mu/2}[K_0\{\tfrac{1}{2}m(u^2+v^2)^{\frac{1}{2}}\} - u(u^2+v^2)^{-\frac{1}{2}}K_1\{\tfrac{1}{2}m(u^2+v^2)^{\frac{1}{2}}\}]\times$$
$$\times\, Q(X-u, Z-v)\,du\,dv\,. \tag{5.7}$$

This result should reduce to (4.8) when $Q(X, Z)$ is independent of Z and provide a useful check on (5.7). It follows in this case that

$$H'(X) = \frac{m^2}{4\pi}\int_{-\infty}^{\infty} Q(X-u)e^{mu/2}\,du\int_{-\infty}^{\infty}[K_0\{\tfrac{1}{2}m(u^2+v^2)^{\frac{1}{2}}\}+$$
$$-u(u^2+v^2)^{-\frac{1}{2}}K_1\{\tfrac{1}{2}m(u^2+v^2)^{\frac{1}{2}}\}]\,dv\,,$$
$$= \frac{m^{\frac{3}{2}}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} Q(X-u)e^{mu/2}(|u|^{\frac{1}{2}} - u/|u|^{\frac{1}{2}})K_{\frac{1}{2}}(\tfrac{1}{2}m|u|)\,du\,, \tag{5.8}$$

using an integral formula listed by Gradshteyn and Ryzhik [6], p. 705. Since the Bessel function $K_{\frac{1}{2}}$ is an elementary function, equation (5.8) reduces to

$$H'(X) = m\int_{-\infty}^{0} Q(X-u)e^{mu}\,du\,,$$

which is equivalent to (4.8).

The integrals in (5.3) or (5.7) can be evaluated in some simple cases. For example, if

$$q = A\cos(\omega x)\cos(\Omega z)\,,$$

and we choose $q_0 = A$ and $x_0 = 1/\omega$, then $Q = \cos X\cos(\mu Z)$ where $\mu = \Omega/\omega$. The double Fourier transform of Q expressed in terms of generalised functions becomes

$$\bar{Q} = \pi^2\{\delta(r-1)+\delta(r+1)\}\{\delta(s-\mu)+\delta(s+\mu)\}\,,$$

where $\delta(x)$ is the Dirac delta function. Inversion of this transform produces

$$H' = m(m\cos X - (1+\mu^2)\sin X)\cos(\mu Z)/\{(1+\mu^2)^2+m^2\}\,.$$

This forced wave has the same periods still in the X- and Z-directions, a decreased amplitude and a phase shift in the X-direction, the crest preceding that of the forced wave as in two-dimensional case.

Further solutions can be obtained in closed form if the forcing term is harmonic in either X or Z.

6. Comment

Some useful comparisons can be made with the paper by S. H. Smith [7] which appears in this Journal. The intuitive approach adopted above is complemented by the more formal derivation given in [7], and both analyses lead to essentially the same equation for the free surface elevation given by equation (14) in [7] and equation (4.2) here. Numerical solutions of this equation are given in both papers: in [7] for isolated humps and in Section 4 for a periodic disturbance.

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